

MAT 534: HOMEWORK 6
DUE W OCT 7

Problems marked by asterisk (*) are optional.

- *1.** Let $a_1, \dots, a_k \in \mathbb{Z}$ be such that $(a_1, \dots, a_k) = 1$ (the gcd). Prove that there is $k \times k$ matrix M with integer coefficients and determinant 1 such that its first row is $a_1, \dots, a_k \in \mathbb{Z}$.
- *2.** Using previous problem, prove that every torsion-free finitely generated abelian group (i.e., the group without elements of finite order) is free.
- *3.** Prove that $\langle a, b \mid a^2 = 1, b^3 = 1, aba = b^2 \rangle = S_3$.
- *4.** Let S be a finite set, $|S| = n$, and $w_1, \dots, w_k$ — words from the alphabet $S \cup \bar{S}$. Prove that there is a group G with generators $x_1, \dots, x_n$ and relations $r_1 = \dots = r_k = 1$, where r_j is obtained from w_j by replacing s_i by x_i and $\bar{s}_i$ by x_i^{-1} , having the following *universality property*: for every group G' on n generators with these relations there is a surjective homomorphism $f : G \rightarrow G'$.
- 5.** Which of the following rings are fields? integral domains? In each case, find all units (invertible elements).
 - (a) $R = \mathbb{F}[x]$, where F is a field.
 - (b) $R = \mathbb{Q}[\sqrt{d}]$, where $d \in \mathbb{Z}$ is square-free.
 - (c) $R = \mathbb{Z}[\sqrt{d}]$, where $d \in \mathbb{Z}$ is square-free.
 - (d) $R = \mathbb{R}[A] \subset M_{2 \times 2}(\mathbb{R})$, where $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.
- 6.** Let F be a field and $\mathbb{F}[[x]]$ be the set of all formal power series in variable x with coefficients in $\mathbb{F}$. Prove that $\mathbb{F}[[x]]$ is a ring, and that $a_0 + a_1x + a_2x^2 + \dots$ is a unit in this ring iff $a_0 \neq 0$.